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New integrable problems of motion of a particle in the plane under the action of potential and conservative zero-potential forces

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Abstract. We consider the general problem of plane motion of a charged particle under the action of potential forces in the same plane and a magnetic field orthogonal to it. Four new integrable cases are pointed out in which the gyroscopic (zero-potential) forces play an essential role. Physical interpretation is given for some of the results in terms of motion under potential and Lorentz' forces in the rotating plane.

1. Introduction

Consider the problem of plane motion of a particle of unit mass under the action of forces of velocity dependent potential characterized by the Lagrangian

$$L = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) + l_1\dot{x} + l_2\dot{y} - V \tag{1}$$

where V, l_1, l_2 depend only on the coordinates x, y. The equations of motion can be written in the canonical variables using the Hamiltonian

$$H = \tfrac{1}{2}[(p_x - l_1)^2 + (p_y - l_2)^2] + V. \tag{2}$$

However, the Lagrangian equations of motion in their explicit form

$$\ddot{x} = -\Omega\dot{y} - \frac{\partial V}{\partial x} \qquad \ddot{y} = \Omega\dot{x} - \frac{\partial V}{\partial y} \tag{3}$$

have the advantage of involving only the two functions $\Omega = \frac{\partial l_1}{\partial y} - \frac{\partial l_2}{\partial x}$ and V. For this reason we shall characterize our system by those two functions.

System (3) admits the Jacobi integral

$$I_1 = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) + V = h. \tag{4}$$

That is the Hamiltonian (2) written in the Lagrangian variables. For this system to be integrable, or for the system described by the Hamiltonian (2) to be completely integrable in the sense of Liouville, a second integral of motion I_2 functionally independent of I_1 must be found.

When $\Omega \equiv 0$ we can choose $l_1 = l_2 = 0$. The equations of motion become time reversible. Only this version of the problem has been intensively studied over the past three decades for integrability. The search resulted in a large list of integrable cases [1]. The second integral was usually assumed a polynomial in the velocities (or momenta) whose degrees range up to

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the sixth. In particular, it was shown that a quadratic integral exists only in the cases when the Hamilton–Jacobi equation is separable in one of the elliptic, parabolic, polar or Cartesian coordinates [2].

Despite its importance in applications the time irreversible case $\Omega \neq 0$ has not been fully studied. Very few cases of existence of a quadratic integral are known but none of higher-degree integrals. Until recently the list of those results comprised only the two cases found by Vandervoort [3] and by Dorizzi *et al* [4]. In our work [6] we have used a method introduced in [5] to construct irreversible mechanical systems with two degrees of freedom, whose configuration is not necessarily the Euclidean plane, and which admit quadratic integrals. This method has proved successful in finding several new integrable problems. It turned out that some of those new systems reduce naturally to plane systems when their parameters are chosen to produce zero Gaussian curvature of their configuration spaces. In this way we have obtained five irreversible plane systems including the previously known two [6].

In this work we introduce four new integrable cases. Some of those cases unify and generalize previously found results of [6]. We also attempt a physical interpretation of some of those cases.

2. Transformation of the equations of motion

It is well known that the system (3) is form-invariant on fixed levels of the integral (4) under conformal transformations of the complex z-plane (see, e.g. [5, 6] and the references therein). In fact, the change of variables and time

$$x + \mathrm{i}y = z = z(\zeta) \qquad \zeta = \xi + \mathrm{i}\eta \qquad \mathrm{d}t = \left|\frac{\mathrm{d}z}{\mathrm{d}\zeta}\right|^2 \mathrm{d}\tau \tag{5}$$

transforms (3) to the form

$$\frac{\mathrm{d}^2\xi}{\mathrm{d}\tau^2} = -\tilde{\Omega}\frac{\mathrm{d}\eta}{\mathrm{d}\tau} - \frac{\partial\tilde{V}}{\partial\xi} \qquad \frac{\mathrm{d}^2\eta}{\mathrm{d}\tau^2} = \tilde{\Omega}\frac{\mathrm{d}\xi}{\mathrm{d}\tau} - \frac{\partial\tilde{V}}{\partial\eta} \tag{6}$$

where

$$\tilde{\Omega} = \left|\frac{\mathrm{d}z}{\mathrm{d}\zeta}\right|^2 \Omega \qquad \tilde{V} = \left|\frac{\mathrm{d}z}{\mathrm{d}\zeta}\right|^2 (V - h). \tag{7}$$

The integral (4) transforms to

$$\frac{1}{2}\left[\left(\frac{\mathrm{d}\xi}{\mathrm{d}\tau}\right)^2 + \left(\frac{\mathrm{d}\eta}{\mathrm{d}\tau}\right)^2\right] + \tilde{V} = 0 \tag{8}$$

i.e. the original motion on the Jacobi level h is equivalent to the transformed motion only on the zero level of the Jacobi integral of the latter, while the constant h enters as a parameter in the transformed potential.

We notice, however, an interesting case when the potential V has the structure

$$V = V_0 - h_1 \left|\frac{\mathrm{d}\zeta}{\mathrm{d}z}\right|^2 \tag{9}$$

where h_1 is an arbitrary parameter. In that case

$$\tilde{V} = \bar{V} - h_1 \qquad \bar{V} = \left|\frac{\mathrm{d}z}{\mathrm{d}\zeta}\right|^2 (V_0 - h) \tag{10}$$

and (8) can be written as

$$\tfrac{1}{2}\left[\left(\frac{\mathrm{d}\xi}{\mathrm{d}\tau}\right)^2+\left(\frac{\mathrm{d}\eta}{\mathrm{d}\tau}\right)^2\right]+\bar{V}=h_1. \tag{11}$$

The transformed motion can be considered free of the above restriction and h_1 is just the Jacobi constant for that motion. This situation will be used below to generate general integrable cases from known ones.

3. New integrable cases

3.1. The first case

Let the pair of functions Ω, V in (3) be given by

$$\Omega = 6ar^2+2b \tag{12}$$

$$V = Ax+By-h_1r^2-abr^4-a^2r^6+2a(ar^2+b)[4cdxy+(c^2-d^2)(x^2-y^2)] \tag{13}$$

where $r=\sqrt{x^2+y^2}$ and a, b, A, B, h_1, c, d are arbitrary parameters. Then, in addition to the integral (4), system (3) admits the integral

$$\begin{aligned}I_2 = a[&(x-c)\dot{y}-(y-d)\dot{x}-\tfrac{3}{2}ar^4-2a(dx-cy)^2\\&-b(r^2+d^2+c^2)+2(ar^2+b)(cx+dy)]\\&\times[(x+c)\dot{y}-(y+d)\dot{x}-\tfrac{3}{2}ar^4-2a(dx-cy)^2\\&-b(r^2+d^2+c^2)-2(ar^2+b)(cx+dy)]\\&-h_1[x\dot{y}-y\dot{x}-\tfrac{3}{2}ar^4-2a(dx-cy)^2-br^2]\\&+\tfrac{1}{2}(A\dot{y}-B\dot{x})-2a(Ad-Bc)(dx-cy)-(ar^2+b)(Ax+By)\end{aligned} \tag{14}$$

and is thus integrable.

This case unifies two cases found in sections 5.4 and 6.3 of [6]. It involves two parameters c, d more than the first and three parameters A, B, d more than the second. Note that the pair Ω, V is form-invariant with respect to rotation of the axes at a fixed angle, but two parameters can be added to it by translating the origin.

3.2. The second case

Now we shall apply to the previous case the transformation

$$z=\sqrt{2\zeta} \qquad \mathrm{d}t=\frac{\mathrm{d}\tau}{2\rho} \tag{15}$$

where $\rho=\sqrt{\xi^2+\eta^2}$. We arrive at the form (6) in which

$$\tilde{\Omega}=6a+\frac{b}{\rho}$$

$$\begin{aligned}\tilde{V}-h_1=\bar{V}=&-\frac{h}{2\rho}-2ab\rho-4a^2\rho^2+4a\left(a+\frac{b}{2\rho}\right)[2cd\eta+(c^2-d^2)\xi]\\&+\frac{1}{2\rho}\left(A\sqrt{\rho+\xi}+B\sqrt{\rho-\xi}\right).\end{aligned} \tag{16}$$

This case involving seven parameters a, b, c, d, h, A and B is integrable. According to section 2 it admits Jacobi's integral

$$\tfrac{1}{2}\left[\left(\frac{\mathrm{d}\xi}{\mathrm{d}\tau}\right)^2+\left(\frac{\mathrm{d}\eta}{\mathrm{d}\tau}\right)^2\right]+\bar{V}=h_1. \tag{17}$$

The second integral can be deduced from (14) by enforcing the substitution (15). Note that h_1 appears in the integral (14) as a coefficient before a linear expression in the velocities. The second integral in this case is a second-degree polynomial in velocities on the fixed level h_1 of (17). In four-dimensional state space the second integral is a polynomial of the third-degree in velocities. We do not need to write down the integral in general, since the explicit solution of the equations of motion can be also deduced from that of the preceding case using the relations (15) in the reverse way

$$\begin{aligned} \xi &= \frac{x^2 - y^2}{2} \qquad \eta = xy \\ \tau &= \int (x^2 + y^2)\,\mathrm{d}t. \end{aligned} \tag{18}$$

However, as we can see, both functions $\tilde{V}$ and $\tilde{\Omega}$ exhibit a singularity at the origin when $b \neq 0$. The potential is also double-valued in the plane if at least one of the parameters A, B does not vanish. Moreover, the second integral will be quite complicated and double-valued.

3.2.1. A special case. In the case $A = B = 0$ the potential and the second integral are single-valued. For the sake of clarity we shall rename the physical variables ξ, η, τ as x, y, t and write this case down as an integrable case of (3) in the form:

$$\begin{aligned} \Omega &= 6a + \frac{b}{r} \\ V &= -\frac{h}{2r} + 4a\left(a + \frac{b}{2r}\right)[2cdy + (c^2 - d^2)x] - 2abr - 4a^2r^2. \end{aligned} \tag{19}$$

The linear terms in the potential characterize a uniform field. Without loss of generality we can direct the x-axis in the direction of that field and thus write

$$\Omega = 6a + \frac{b}{r} \qquad V = -\frac{h}{2r} - 2abr - 4a^2r^2 + gx\left(1 + \frac{b}{2ar}\right). \tag{20}$$

The second integral corresponding to this choice is of the third degree and has the form:

$$\begin{aligned} I_2 = &\left(\dot{x}^2 + \dot{y}^2 - \frac{h}{r} - 8a^2r^2 + 2gx\left(1 + \frac{b}{2ar}\right)\right) \\ &\times[4a(x\dot{y} - y\dot{x} - 3ar^2) - g(r - x) - 4abr] \\ &-[4a(x\dot{y} - y\dot{x} - 3ar^2) - g(r - x) - 2abr]^2 \\ &+\frac{g[y\dot{y} - (r - x)\,\dot{x} - 4ayr - 2by]^2}{(r - x)} + 4a^2b^2r^2. \end{aligned} \tag{21}$$

This can be verified directly by the use of the equations of motion.

The case characterized by (20) generalizes the case indicated in [10]. In fact, when we set $b = 0$ (20) reduces to:

$$\Omega = 6a \qquad V = -\frac{h}{2r} - 4a^2r^2 + gx. \tag{22}$$

3.2.2. Physical interpretations of an integrable problem. In [10] case (22) was considered as an approximate model for the problem of the motion of an electron in a hydrogen atom in a circularly polarized microwave field and a static magnetic field orthogonal to the plane of the polarization. For the purpose of potential application of this system in problems of physics and mechanics, we attempt two different interpretations of the integrable system with Ω and V as in (22).

Rotating Stark potential and a rotating magnetic field. Consider a reference system $Oxyz$, which is rotating, with respect to an inertial system $OXYz$, with the uniform angular velocity ω about the common z-axis. Let a particle of unit mass and electric charge q be moving in the plane $z = 0$ under the action of forces of potential $v = v(x, y)$. Moreover, assume that there is a magnetic field H which is, in the rotating system, static, homogeneous and pointing in the z-direction. The equations of motion can be shown to have the form (3) in the rotating frame, where

$$\Omega = -\left(2\omega + \frac{q}{c}H\right) \qquad V = v(x, y) - \tfrac{1}{2}\omega^2 r^2 \tag{23}$$

where c is the speed of light (for a discussion of the Lorentz force see e.g. [9]).

To identify those expressions with (22) we take

$$a = -\frac{\omega}{2\sqrt{2}} \tag{24}$$

and

$$v = -\frac{h}{2r} + gx \qquad \frac{q}{c}H = \left(\frac{3\sqrt{2}}{2} - 2\right)\omega. \tag{25}$$

This means that the problem characterized by (23) is integrable for the choice (25). The potential in the rotating plane is the Stark potential composed of a central Newtonian (or Coulomb) term and a uniform field term. This problem is known to be integrable when $\omega = 0$ (see, e.g., [7]). If, however, a uniform rotation is imposed, it becomes non-integrable [8]. Our result asserts that the rotating Stark problem becomes integrable (with quadratic second integral) if we add a magnetic field orthogonal to the plane and rotate it. It is interesting to note that the ratio of the Lorentz force to the Coriolis force is $\frac{\frac{q}{c}H}{2\omega} = \frac{3}{4}\sqrt{2} - 1 = 0.060\,66$.

Rotating Stark potential and a static homogeneous magnetic field. Consider a situation that differs from the one just described only in that the magnetic field H is static in the inertial frame $OXYz$. The equations of motion in the rotating frame can be shown to have the form (3), where

$$\Omega = -\left(2\omega + \frac{q}{c}H\right) \qquad V = v(x, y) - \frac{1}{2}\omega\left(\omega + \frac{q}{c}H\right)r^2. \tag{26}$$

Note that in this case, unlike the previous one, the rotation has transformed the Lorentz force and invoked a central potential term $-\frac{q\omega H}{2c}r^2$, which has the structure of a centrifugal or centripetal force, depending on the signs of the parameters q, ω and H.

To identify those expressions with (22) we can take one of the choices:

$$a = -\frac{\omega}{2}$$

and

$$v = -\frac{h}{2r} + gx \qquad \frac{q}{c}H = \omega \tag{27}$$

or

$$a = -\frac{\omega}{4}$$

and

$$v = -\frac{h}{2r} + gx \qquad \frac{q}{c}H = -\frac{\omega}{2}. \tag{28}$$

The problem characterized by (26) is integrable under either (27) or (28). Note that the ratio $\frac{\frac{q}{c}H}{2\omega} = \frac{1}{2}, -\frac{1}{4}$ for the two choices, respectively.

3.3. The third case

For the choice

$$\begin{aligned}\Omega &= 6ax \\ V &= Ax + By - a(x^2 + y^2)[C + \tfrac{1}{2}a(5x^2 + y^2)] + 4a^2bx^3\end{aligned} \tag{29}$$

equations (3) admit the second integral

$$\begin{aligned}I_2 = a\{\dot{y} - a(y^2 + 3x^2) - C\}\{(x - b)\dot{y} - y\dot{x} - a[2x^3 - b(3x^2 - y^2) + 2xy^2] - bC\} \\ +\tfrac{1}{2}A[\dot{y} - a(y^2 + 3x^2)] - \tfrac{1}{2}B[\dot{x} + 2ay(x + 2b)]\end{aligned} \tag{30}$$

and are thus integrable.

This case contains one parameter b more than that presented in section 6 of [6]. Note that three more parameters can be added to the system (29), (30) by displacing the origin and rotating the axes at a fixed angle about the new origin.

As in the previous subsection, we can give a physical interpretation of the present case. A gyroscopic coefficient $6ax$ can be interpreted as a (nonuniform) magnetic field orthogonal to the plane of motion. In fact, the three-dimensional scalar harmonic potential $V_m = -6axz$ gives rise to the magnetic field $H = 6a(z, 0, x)$, whose restriction to the plane $z = 0$ is $(0, 0, 6ax)$. Similarly, the potential V in (29) is the plane restriction of the harmonic potential

$$\begin{aligned}V_{3d} = Ax + By + C(2z^2 - x^2 - y^2) + 4a^2b(x^3 - 3xz^2) \\ +a^2[-\tfrac{1}{2}(x^2 + y^2)(5x^2 + y^2) + 6z^2(3x^2 + y^2) - 4z^4].\end{aligned}$$

We do not claim, however, that the three-dimensional problem is integrable. To decide that further investigation is needed.

It is noteworthy that system (29) can be used to generate another general integrable problem through the application of the transformation (15). The resulting potential is multivalued and the integral becomes of the third degree in the velocity.

3.4. The fourth case

This case results from the special case of section 6.2.1 of [6] when the Gaussian curvature of the configuration manifold vanishes. If in (3) we take

$$\begin{aligned}V &= \frac{C\alpha(\alpha - a)}{(a - \alpha)x^2 - \alpha y^2 + \alpha a(\alpha - a)} \\ \Omega &= \frac{1}{2}C_1(a - \alpha)\left[\frac{\alpha}{(a - \alpha)x^2 - \alpha y^2 + \alpha a(\alpha - a)}\right]^{3/2}\end{aligned} \tag{31}$$

we shall have the second integral

$$\begin{aligned}I_2 = [(x - a)\dot{y} - y\dot{x}][(x + a)\dot{y} - y\dot{x}] - \frac{C_1[\alpha y\dot{x} + x(a - \alpha)\dot{y}]}{\sqrt{(a - \alpha)x^2 - \alpha y^2 + \alpha a(\alpha - a)}} \\ +\frac{\alpha[8aCy^2 - C_1(a - \alpha)]}{4[(a - \alpha)x^2 - \alpha y^2 + \alpha a(\alpha - a)]}.\end{aligned} \tag{32}$$

The equipotentials have the form of the conic sections

$$\frac{x^2}{\alpha a} + \frac{1}{a(\alpha - a)}y^2 = 1 - \frac{C}{Va}.$$

The potential is finite in the whole plane only when $\alpha < 0$. When $\alpha > 0$, the potential has a singular line

$$\frac{x^2}{a\alpha} + \frac{1}{a(\alpha - a)}y^2 = 1$$

which is a hyperbola for $\alpha < a$ and an ellipse for $\alpha > a$. In the last case, the motion of the particle is possible only on one side of the singular line.

When $C_1 = 0$, $\Omega = 0$ and the system becomes time reversible and separable in elliptic coordinates in the plane.

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